Short Course on Experimental Dynamic Substructuring

Module #4: General theory (Part 2)

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Outline

- Coupling the physical model
  - 3-field form
  - Primal assembly
  - Dual assembly
  - Summary & Remarks
- Coupling in the frequency domain
  - Assembled admittance
  - Example
  - Summary & Remarks
- **Dual coupling in representation space**
- **Weak interface compatibility**
  - By projection
  - By filtering
- **Summary**

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Outline

Dual coupling in representation space (Modal)

- How do the assembled equations look if

each substructure is approximated by a
reduced set of (modal) shapes stored in the columns of $\boldsymbol{R}^{(s)}$

$$\boldsymbol{u}^{(s)} \simeq \boldsymbol{R}^{(s)}\eta^{(s)}$$

in local equilibria

$$\boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)}(t) = \boldsymbol{f}^{(s)}(t) + \boldsymbol{g}^{(s)}(t)$$

$$\boldsymbol{M}^{(s)}\boldsymbol{R}^{(s)}\ddot{\eta}^{(s)} + \boldsymbol{C}^{(s)}\boldsymbol{R}^{(s)}\dot{\eta}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{R}^{(s)}\eta^{(s)} = \boldsymbol{f}^{(s)} + \boldsymbol{g}^{(s)} + \boldsymbol{r}^{(s)}$$

Equilibrium error due to response approximation

or in block notations

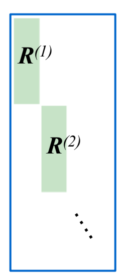
$$\boldsymbol{u} \simeq \boldsymbol{R}\eta$$

$$\boldsymbol{R} \triangleq \text{diag}\left(\boldsymbol{R}^{(1)}, \ldots, \boldsymbol{R}^{(n)}\right)$$

in local equilibria

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g}$$

$$\boldsymbol{M}\boldsymbol{R}\ddot{\eta} + \boldsymbol{C}\boldsymbol{R}\dot{\eta} + \boldsymbol{K}\boldsymbol{R}\eta = \boldsymbol{f} + \boldsymbol{g} + \boldsymbol{r}$$



Dual coupling in representation space (Modal)

$$\boldsymbol{u} \simeq \boldsymbol{R}\eta \qquad \boldsymbol{M}\boldsymbol{R}\ddot{\eta} + \boldsymbol{C}\boldsymbol{R}\dot{\eta} + \boldsymbol{K}\boldsymbol{R}\eta = \boldsymbol{f} + \boldsymbol{g} + \boldsymbol{r}$$

Equilibrium error due to response approximation

Asking that the residual force creates no work in the approximation space
(as done in reduction techniques – virtual work),

$$\boldsymbol{R}^T\left(\boldsymbol{M}\boldsymbol{R}\ddot{\eta} + \boldsymbol{C}\boldsymbol{R}\dot{\eta} + \boldsymbol{K}\boldsymbol{R}\eta = \boldsymbol{f} + \boldsymbol{g} + \cancel{\boldsymbol{r}}\right)$$

$$\Rightarrow \boxed{\boldsymbol{M}_m\ddot{\eta} + \boldsymbol{C}_m\dot{\eta} + \boldsymbol{K}_m\eta = \boldsymbol{f}_m + \boldsymbol{g}_m}$$

where
$$\left\{\begin{array}{lcl} \boldsymbol{M}_m & \triangleq & \boldsymbol{R}^T\boldsymbol{M}\boldsymbol{R} \\ \boldsymbol{C}_m & \triangleq & \boldsymbol{R}^T\boldsymbol{C}\boldsymbol{R} \\ \boldsymbol{K}_m & \triangleq & \boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R} \\ \boldsymbol{f}_m & \triangleq & \boldsymbol{R}^T\boldsymbol{f} \\ \boldsymbol{g}_m & \triangleq & \boldsymbol{R}^T\boldsymbol{g} \end{array}\right.$$
are the matrices in the representation space (reduced)

The compatibility condition $\boldsymbol{B}\boldsymbol{u} = \mathbf{0}$ transforms into $\boxed{\boldsymbol{B}_m\eta = \mathbf{0}}$

where $\boxed{\boldsymbol{B}_m \triangleq \boldsymbol{B}\boldsymbol{R}}$

Dual coupling in representation space (Modal)

Following the same reasoning as for the general case, one can
write the dually assembled problem in the representation space [1]

$$\boxed{\begin{bmatrix} \boldsymbol{M}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \dot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_m & \boldsymbol{B}_m^T \\ \boldsymbol{B}_m & \mathbf{0} \end{bmatrix}\begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{0} \end{bmatrix}}$$

REMARKS:

- When expressed in the frequency domain, the assembled problem in representation space is

$$\boxed{\begin{bmatrix} \boldsymbol{Z}_m & \boldsymbol{B}_m{}^T \\ \boldsymbol{B}_m & \mathbf{0} \end{bmatrix}\begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{0} \end{bmatrix}} \quad \text{where} \quad \boldsymbol{Z}_m = \boldsymbol{R}^T\boldsymbol{Z}\boldsymbol{R}$$

- Looks very similar to the Dual problem in the physical space but …
  - … the matrices are reduced and expressing the physical properties in the representation space
  - … the unknowns are now the amplitudes of the representation vectors

- It is important to build the reduction space $\boldsymbol{R}$ on shapes that can properly represent the response of the *assembled* system
  → some methods use modes of the system when assembled with a dummy mimicking roughly neighboring structures (see the transmission simulator idea in MOD9)

Dual coupling in representation space (Modal)

$$\begin{bmatrix} \boldsymbol{M}_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \ddot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}_m & \boldsymbol{B}_m \\ \boldsymbol{B}_m^T & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_m \\ \mathbf{o} \end{bmatrix}$$

REMARKS (continued):

- If the approximation space is the modal space of the substructures (free interface),

$$\boldsymbol{u}^{(s)} = \boldsymbol{\Phi}^{(s)} \eta^{(s)} \quad \text{where} \quad \boldsymbol{K}^{(s)} \boldsymbol{\Phi}^{(s)} = \boldsymbol{M}^{(s)} \boldsymbol{\Phi}^{(s)} \boldsymbol{\Omega}^{(s)^2}$$

  then in frequency domain (with proportional damping)

$$\begin{bmatrix} \begin{bmatrix} -\omega^2 \boldsymbol{I} + i\omega 2\zeta^{(1)} \boldsymbol{\Omega}^{(1)} + \boldsymbol{\Omega}^{(1)^2} & 0 \\ 0 & \ddots \end{bmatrix} & \begin{bmatrix} \boldsymbol{\Phi}^{(1)^T} \boldsymbol{B}^{(1)^T} \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{B}^{(1)} \boldsymbol{\Phi}^{(1)} & \cdots \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \eta^{(1)} \\ \vdots \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^{(1)^T} \boldsymbol{f}^{(1)} \\ \vdots \\ 0 \end{bmatrix}$$

- One major issue: the condition $\boldsymbol{B}\boldsymbol{R}\eta = \boldsymbol{B}_m \eta = \mathbf{o}$ requires full compatibility on the interface.
  Since each substructures is represented by a small numbers of local modes,
  this might lead to ***locking*** (the compatibility can be satisfied only when 0 response)
  → ***need for weakening the interface compatibility***

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Weak interface compatibility

Enforcing the strong compatibility for the entire interface can lead to serious problems when (see earlier discussions) ….

- the admittance contains measurement errors and the interface is stiff

- when the substructures are approximated by a limited number of representative shapes that do not necessarily match on the interface (locking)

➡ 2 different techniques to relax the interface compatibility (***weakening***)

1. requiring that only a projection of the interface gap is 0
   → **Projected compatibility**

2. projecting the interface displacements on each side of the interface on a common reduced space and enforcing compatibility only for that space
   → **Filtering**

Weak interface compatibility: Projected Compatibility

Starting from the frequency representation with physical dofs (would be similar for modal representation):

$$\boxed{\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}}$$

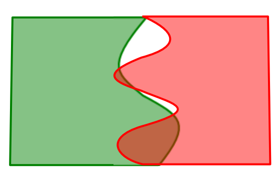
In order to weaken the compatibility condition, we will require only that

the projection of the interface gap on a small number of interface modes is 0

$$\boxed{\boldsymbol{\Gamma}_I^T \left(\boldsymbol{B}\boldsymbol{u}\right) = 0}$$

where $\boldsymbol{\Gamma}_I$ contains the interface projection shapes.

For instance if $\boldsymbol{\Gamma}_I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, ….



….. one imposes only an average compatibility

Weak interface compatibility: Projected Compatibility

One can show that the dual problem then writes

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T\boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T\boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{0} \end{bmatrix}$$

- small number of compatibility conditions on the interface (projected)
- small number of interface force variables λ_Γ
- can be interpreted as if the compatibility is enforced through interface forces that are built on the interface projection shapes: $\lambda \simeq \Gamma_I \lambda_\Gamma$

- Similarly, if the substructure displacements are approximated in a representation space:

$$\boldsymbol{u} \simeq \boldsymbol{R}\eta \qquad \text{and} \qquad \begin{bmatrix} \boldsymbol{Z_m} & \boldsymbol{B_m}^T\boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T\boldsymbol{B_m} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_m} \\ \mathbf{0} \end{bmatrix}$$

in which the displacements ***and*** the interface forces are approximated.

Weak interface compatibility: Filtering

On the second approach, one fits the interface displacement on both sides of the interface on a common displacement shape.

Calling $\boldsymbol{\Phi}_\Gamma$ the assumed modes for the interface displacements,

$$\boldsymbol{u}^{(s)} = \begin{bmatrix} \boldsymbol{u}_i^{(s)} \\ \boldsymbol{u}_c^{(s)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_i^{(s)} \\ \underbrace{\boldsymbol{\Phi}_\Gamma \alpha^{(s)}}_{\text{common interface modes}} + \underbrace{\tilde{\boldsymbol{u}}_c^{(s)}}_{\text{part not representable in } \boldsymbol{\Phi}_\Gamma} \end{bmatrix} \begin{array}{l} \rightarrow \textit{internal dofs} \\ \rightarrow \textit{connecting dofs} \end{array}$$

One will enforce a weak compatibility by asking only that
the amplitudes of $\boldsymbol{\Phi}_\Gamma$ should be equal on each side of the interface: $\boldsymbol{B}\alpha = 0$

$$\boldsymbol{u}_c^{(s)} \simeq \boldsymbol{\Phi}_\Gamma \alpha^{(s)}$$

and by least square

$$\alpha^{(s)} = \left(\boldsymbol{\Phi}_\Gamma^T \boldsymbol{\Phi}_\Gamma\right)^{-1} \boldsymbol{\Phi}_\Gamma^T \boldsymbol{u}_c^{(s)}$$

$$\Rightarrow \quad \boldsymbol{B}\left(\boldsymbol{\Phi}_\Gamma^T \boldsymbol{\Phi}_\Gamma\right)^{-1} \boldsymbol{\Phi}_\Gamma^T \boldsymbol{u}_c = \boldsymbol{B}\underbrace{\boldsymbol{F}}_{\text{filter that removes everything that is not in the space of } \boldsymbol{\Phi}_\Gamma}\boldsymbol{u} = 0$$

Weak interface compatibility: Filtering

Finally the dual problem writes

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{F}^T\boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

- similarly if the displacements are approximated in a representation space

$$\begin{bmatrix} \boldsymbol{Z_m} & \boldsymbol{R}^T\boldsymbol{F}^T\boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F}\boldsymbol{R} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_m} \\ \boldsymbol{0} \end{bmatrix}$$

- λ_α has the meaning of generalized forces associated to the interface displacements
- An important application: when $\boldsymbol{\Phi}_\Gamma$ contains the rigid body motion of the interface, then λ_α are the 6 forces and moments resultants at the interface
  - → often used for rigid interfaces (bolts, brackets ….), allowing to construct rotational dofs
  - → in that case the method is also called EMPC (Explicit Multi-Point connection) or virtual point transformation [11,12]

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Summary

$$\boxed{\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}} \Leftrightarrow \boxed{\boldsymbol{u} = \overbrace{\left(\boldsymbol{Y} - \boldsymbol{Y}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^T)^{-1}\boldsymbol{B}\boldsymbol{Y}\right)}^{\text{assembled admittance}}\boldsymbol{f}}$$

or

$$\begin{bmatrix} \boldsymbol{Z_m} & \boldsymbol{B_m}^T \\ \boldsymbol{B_m} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_m} \\ \boldsymbol{0} \end{bmatrix}$$

in reduced representation space $\boldsymbol{u} \simeq \boldsymbol{R}\eta$

$$\boldsymbol{Z}_m = \boldsymbol{R}^T \boldsymbol{Z} \boldsymbol{R}$$

Weakening of compatibility by *projection*:

$$\lambda \simeq \Gamma_I \lambda_\Gamma$$

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{B}^T \boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

or

$$\begin{bmatrix} \boldsymbol{Z_m} & \boldsymbol{B_m}^T \boldsymbol{\Gamma}_I \\ \boldsymbol{\Gamma}_I^T \boldsymbol{B_m} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_m} \\ \boldsymbol{0} \end{bmatrix}$$

Weakening of compatibility by *filtering*:

$$\boldsymbol{u}_c^{(s)} \simeq \boldsymbol{\Phi}_\Gamma \alpha^{(s)}$$

$$\boldsymbol{F} = \left(\boldsymbol{\Phi}_\Gamma^T \boldsymbol{\Phi}_\Gamma\right)^{-1} \boldsymbol{\Phi}_\Gamma^T$$

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{F}^T \boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}$$

or

$$\begin{bmatrix} \boldsymbol{Z_m} & \boldsymbol{R}^T \boldsymbol{F}^T \boldsymbol{B}^T \\ \boldsymbol{B}\boldsymbol{F}\boldsymbol{R} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_m} \\ \boldsymbol{0} \end{bmatrix}$$

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